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# Some New Methods in Kinematical Theory

BY

C. V. RAMAN

In the course of my recent experimental work on the vibrations of bowed strings,[*] it occurred to me that the ordinary Fourier analysis which has been used by previous investigators[†] is not a convenient or suitable method of considering the kinematics of this class of vibrations. In fact, the nature of the case is such that the Fourier analysis obscures instead of elucidating the physical processes set up in the string by the action of the bow. I have therefore developed a new method of treatment which has the great advantage of enabling the subject to be considered entirely from first principles *i.e.*, without any appeal to experiment except for confirming the fully-worked-out predictions of theory. I am using this method in a monograph on the 'Mechanics of Bowed Strings' which is under preparation and which I intend to contribute to one of the regular periodicals for physics. The brief account of this method which I propose to give here may perhaps be of some interest to mathematicians.

The general solution of the equation of wave-propagation on an infinite string not subject to damping is

$$Y=f\,(x-at)\;+\;F\,(x+at)\ldots\ldots\ldots\ldots\ldots(1)$$

It is well-known that this solution for the case of an infinite string can be used to represent the configuration at any instant of a vibrating string of finite length by arranging the form of the *displacement* waves in such manner that the motion is periodic and satisfies the terminal condition $y=o$ at the two ends of the string.

Similarly, the solution obtained by differentiating (1) with respect to time, *viz*

$$\frac{dy}{dt}=\;-\;a\;f'\;(x-at)\;+\;a\;F'\;(x+at)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

can be applied to represent the *velocity-diagram* of a finite string at any instant during its vibration, if the periodicity of the motion and the terminal conditions of velocity are secured. It is obvious that solution (2), as it stands, represents the *velocity-waves* that travel on an infinite string without change of form in the positive and negative directions respectively. In the case of a finite string of length $l$, the reflexions that take place at the two ends have to be taken into account, and we may write

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[*] *Bulletin No. XI of the Indian Association for the Cultivation of Science*, pp. 43-52, May 1914. Also, 'Nature', page 622, August 1914, and Science Abstracts, Fabruary 1915.

[†] Helmholtz, 'Sensations of Tone' English Translation by Ellis, Chapter V and Appendix VI.

Krigar-Menzel and Raps, Über Saitenschwingungen, *Sitzungberichte* of the Berlin Academy, 1891.

A. Stephenson, "On the Maintenance of Periodic Motion by Solid Friction," *Philosophical Magazine*, January 1911.

$$\frac{dy}{dt} = \theta\,(x-at) + \phi\,(x+at) \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The two functions $\theta\,(x-at)$ and $\phi\,(x+at)$ represent the velocity-waves which must be imagined as extending to infinity in both directions, and as being perfectly periodic with wave-length equal to twice the length of the string. To satisfy the terminal conditions $\frac{dy}{dt}=o$, we must assume that the positive wave from $x=o$ up to $x=l$ in its initial position is an inverted and reflected image of the negative wave from $x=l$ up to $x=2\,l$, and vice-versa.

The next step in the argument is to consider the changes of velocity that take place at individual points on the string. Obviously the form of the positive and negative velocity waves must be such that by their movement and superposition they reproduce the changes of velocity at any given point on the string.

If now, some point on the string (say the point $x=x_b$) has the characteristic property of always moving with a succession of constant velocities during each period of vibration, the velocity passing in a discontinuous manner from each value to the next, we must have, at that point, the condition $\frac{d^2y}{dt^2}=o$ always satisfied, except at certain instants in each period of vibration when it be comes ± infinity. Differentiating (3) with respect to time, we have

$$\frac{d^2y}{dt^2} = -a\ \theta'(x-at) + a\phi'(x+at) \ldots\ldots\ldots\ldots\ldots (4)$$

Since, at the point $x_b$, $\frac{d^2y}{dt^2}$ is generally zero, we must have

$$\theta'\ (x_b-at) = \phi'\ (x_b+at) \ldots\ldots\ldots\ldots\ldots (5)$$

If the velocity-waves $\theta\,(x-at)$ and $\phi\,(x+at)$ are represented graphically, equation (5) may be given a geometrical significance. If any two points are taken, one on the positive wave and one on the negative wave, the distances of which from the point $x_b$ measured along the string are equal but in opposite directions, we should find the slopes of the waves at the two points to be equal. As already mentioned, the form of the velocity-waves must satisfy certain other conditions, *viz.*, that they are periodic with wave-length $2l$, and that initially the form of the positive wave from $x=o$ up to $x=l$ is an inverted and reflected image of the negative wave from $x=l$ up to $x=2l$, and *vice-versa.* It is a simple geometrical problem to find the form of the positive and negative waves which would simultaneously satisfy these three conditions. By inspection, the following remarkably simple and significant solution is obtained: if the point $x_b$ divides the string in an *irrational* ratio, the only possible form of the velocity-waves is that in which the slope is everywhere the same, *i.e.* they are representable by a number of straight lines that are all parallel to one another, a discontinuity intervening wherever one straight line leaves off and the next begins.

The next step in the argument is to show that the *modus operandi* of the bow requires that $\frac{d^2 y}{dt^2}$ should be generally zero at the bowed point except at certain instants in each period of vibration when it becomes $\pm$ infinity. This can be proved from dynamical considerations, for which I must refer the reader to my forthcoming monograph, and it is also shown there that the velocity at the bowed point must alternate between two and only two constant values. The preceding theory is thus applicable when the velocity-waves travel on the string without any appreciable change of form, and the discussion shows the form of the velocity-waves to be representable by a number of parallel straight lines with intervening discontinuities, when the bow is applied at a point dividing the string in a rational ratio, as well as in the cases in which it divides the string in an irrational ratio. The condition that the velocity at the bowed point alternates between two and only two constant values is then used to find the form of the velocity-waves. When the bowed point divides the string in an irrational ratio, the discontinuities in the velocity-waves are all numerically equal in magnitude to one another, and to the arithmetical sum of the two speeds possible at the bowed point. The types of vibration may then be classified according to the *number* of the discontinuities (one, two, three or more) per wave-length, in each wave. In the detailed discussion of these cases, the conditions under which they are excited and the kinematical relations involved therein are investigated, and the modifications that occur when the bow is applied at a *rational* point are also worked out. The general procedure adopted is very simple. If there are $n$ equal discontinuities in each velocity-wave, the lines in the velocity-graph of the string being parallel to one another must evidently all pass through the nodes of the *nth* harmonic or the ends of the string (there being situated at equal intervals along the string). The position of the intervening discontinuities is, in general, arbitrary. From this we get at once, the general kinematical relation

$$w = \frac{n x_n}{l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where $w$ is the ratio of the time during which the bowed point moves with the larger of the two speeds, to the total period of a complete vibration, $n$ is an integer, and $x_n$ is the distance of the bowed point from the *nearest* node of the *nth* harmonic. This relation holds good both for 'rational' and 'irrational' points of application of the bow. When $n=1$, we have the well-known kinematical relation $w=\frac{x}{l}$ discovered by Helmholtz. The relation has been verified by me for the other cases in which $n=2$, 3, or 4 etc.

It is impossible here to enter into further detail of the various developments of the theory outlined above, and I must therefore content myself with summarising briefly, the main results of the research which are as follows :—

(*a*) The theory gives a completely satisfactory account of the observed types of vibration, proceeding from first principles and including the so-called 'complicated' types of motion.

(*b*) It predicts the general kinematical relation (6) given above, of which only the first case ($n=1$) was known previously through the work of Helmholtz. The general relation has since been verified by me experimentally.

(*c*) It predicts the effect to be observed by varying the pressure or velocity of bowing at any given point, rational or irrational, and has an important bearing on the musical applications of the subject.

(*d*) The actual form of the vibration-curve for any given point on the string for any one of the possible or actual types of vibration can be readily traced by a very simple graphical process which dispenses entirely with the tedious methods of harmonic analysis, and the curve thus traced from purely theoretical principles, can be compared directly with that observed in experiment. A large number of examples will be given in my complete monograph.

(*e*) Many of the conclusions arrived at by Krigar-Menzel and Raps as the result of their work require to be largely modified.

(*f*) The whole treatment gives a far more vivid idea of the kinematics of bowed strings than can possibly be conveyed by the Fourier analysis.

# On Sound-waves due to prescribed vibration on a spherical surface in the presence of a rigid and fixed spherical obstacle

BY

SUDHANSUKUMAR BANERJI

In this paper I propose to give the complete solution of the problem of aerial disturbance due to any prescribed vibration on the surface of a sphere in the presence of a rigid and fixed spherical obstacle. This problem remained unsolved up to this time, although, so early as 1868, Sir George Stokes* solved the problem of the propagation of waves outwards from a spherical surface in an unlimited medium, in 1862, Clebsch† investigated the scattering of sound-waves due to a simple point-source placed at a finite distance by a spherical surface, and in 1872, Lord Rayleigh** solved a similar problem of disturbance produced by a spherical surface on plane waves of sound. All the results given in this paper may therefore be regarded as new.

The method adopted may be called the method of successive reflections and has some analogy to the method of images used in solving the problem of the motion of two spheres in an infinite liquid.

I should like to express my indebtedness to Dr. Ganesh Prasad at whose suggestion I took up, and under whom I carried on, the investigation the results of which are embodied in this paper.

## § 1.

For the sake of definiteness, I shall first consider the case when the two spheres are external to each other and the prescribed vibration given to one of them is symmetrical about the line joining the two centres.

Let and B be the two spheres and $a$ and $b$ their radii.

The polar co-ordinates of any external point P are $(r, \theta, \phi)$ referred to the centre of the sphere A, and $(r', \theta', \phi)$ referred to the centre of the sphere B, $\theta$ and $\theta'$ being measured in opposite senses from the line A B.

---

* Stokes, On the communications of vibrations from a vibrating body to a surrounding gas, *Phil. Trans.*, 1868. [*Math. and Phys. Papers, Vol. IV, p.* 299.]

†Clebsch, Über die Reflexion an einer Kugelfläche, *J. f. Math., Bd.* 61, *p.* 195, (1863).

** Lord Rayleigh, Investigation of the disturbance produced by a spherical obstacle on the waves of sound, *Proc. Lond. Math. Soc., Vol. IV, p.* 253, (1872). [*Theory of sound, Art.* 334.]

Since the prescribed vibration on the surface of the sphere A (say) is symmetrical about the line joining the two centres, it must be expressible by a series of the type

$$\sum_{n=o}^{n=m} A_n \, P_n \, (cos\theta). \; e^{ikct},$$

where the $A_n$'s are known constants, and $m$ may be finite or infinite.

## § 2.

If we ignore the presence of the spherical obstacle B in the medium, then there is nothing to obstruct the wave motion started by the sphere A and the problem reduces to the one whose complete solution was obtained by Sir George Stokes* in 1868.

If $\psi_o$ denote the velocity potential of this unobstructed wave motion. then $\psi_o$ is given by

$$\psi_o = \sum_{n=o}^{n=m} A_n \frac{\dfrac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}}{\dfrac{d}{da}\left[\dfrac{K_{n+\frac{1}{2}}(ika)}{\sqrt{a}}\right]} P_n \, (cos\theta) \, e^{ikct}.$$

I shall write this in the form

$$\psi_o = \sum_{n=o}^{n=m} A_n^{(o)} \, \frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n \, (cos\,\theta) \; e^{ikct}.$$

## § 3.

I shall now consider the effect of the presence of the spherical obstacle B in the medium.

For this purpose, I shall first regard the dimensions of the vibrating sphere to be very small in comparison with the wave-length, that is, I shall practically regard it as a multiple point-source of sound of the $m$ *th* and lower degrees of complexity and calculate the scattering of the waves produced by the presence of the spherical obstacle B.

The velocity potential of the wave system incident on the surface of the sphere B is of course given by the above expression for $\psi_o$.

---

* Stokes, loc. cit.

Let the velocity potential of the scattered wave system regarded as radiating outwards from the centre of the spherical obstacle B be denoted by $\psi_1$.

Then the function $\psi_1$ has to satisfy the following conditions:—

(i) it must satisfy $\frac{d^2\psi}{dt^2} = c^2\nabla^2\psi$;

(ii) it must satisfy the condition $\frac{d\psi_o}{dr'} + \frac{d\psi_1}{dr'} = o$,

on the surface of the sphere B, *i.e.*, on $r' = b$;

(iii) together with its first and second derivatives it must be finite and continuous everywhere in the medium;

(iv) it must vanish as $\frac{e^{-ikr'}}{r'}$, when $r'$ tends to infinity.

We can therefore assume $\psi_1$ to be given by the series

$$\psi_1 = \sum_{p=o}^{p=\infty} A_p^{(1)} \frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\cos\theta')\, e^{ikct}.$$

This expression satisfies the conditions (i) and (iii). It also satisfies he condition (iv) for the reason that when $r$ is large

$$K_p(ir) = \sqrt{\frac{\pi}{2r}}\; e^{-i\left(r+\frac{\pi}{4}\right)}$$

approximately.*

## § 4.

The unknown constants $A_p^{(1)}$'s in the expression for $\psi_1$ have to be determined by means of the condition (ii).

If $d$ is the distance between the centres of the two spheres, then

$$r^2 = r'^2 + d^2 - 2r'd\cos\theta'.$$

If we write $\cos\theta = \mu$ and $\cos\theta' = \mu'$, we can always expand the function $\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\mu)$ as an infinite series of functions of the type $\frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\mu')$, provided $r' < d$.

---

* Cf., Nielsen, *Handbuch der Theorie der Cylinderfunktionen*, p. 154.

We can in fact write

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu) = \sum_{p=o}^{p=\infty} A_{n,\,p}\,\frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_p(\mu'),$$

and it is easy to prove that

$$A_{n,\,p} = \frac{(2p+1)\,i^p}{2} \quad \mathop{\mathrm{Lim.}}_{a=o} \int_{-1}^{1} P_n(z)\,P_p(z)\,\phi\,(a,\,d,\,z)\,dz,$$

where

$$\phi\,(a,\,d,\,z) = k\,\sqrt{\frac{2\;\pi\;k}{d}}\,\sum_{m=o}^{m=\infty} e^{-m^2a}\,(m+\tfrac{1}{2})\;K_{m+\frac{1}{2}}(ikd)\;P_m(z).$$

[*The proof can be briefly stated as follows :—*

We have

$$\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}}\;P_n(\mu) = i^{-n}\,\sqrt{\frac{2k}{\pi}}\;P_n\left(\frac{d}{id\,(kx)}\right)\frac{\sin kr}{kr},$$

and $$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\;P_n(\mu) = (-1)^n\,\sqrt{\frac{\pi\,k}{2\,i}}\;P_n\left(\frac{d}{id(kx)}\right)\frac{e^{-ikr}}{kr},$$

the $x$—axis being the line AB.

Also $$\frac{e^{-ikr}}{r} = \frac{2\sqrt{i}}{\sqrt{dr'}}\sum_{m=o}^{m=\infty} i^m\,(m+\tfrac{1}{2})K_{m+\frac{1}{2}}(ikd)\;J_{m+\frac{1}{2}}(kr')\;P_m(\mu'),$$

if $r' < d$.

Therefore

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu) = k\,\sqrt{\frac{\pi k}{2d}}\,\sum_{m=o}^{m=\infty} i^m\,(2\,m+1)\quad P_n\left(\frac{d}{id\,(kx')}\right)$$

$$\times K_{m+\frac{1}{2}}(ikd)\;\frac{J_{m+\frac{1}{2}}(kr')}{\sqrt{r'}}\;P_m(\mu'),$$

where the $x'$-axis is the line BA and has the sense BA, so that $\frac{d}{dx} = -\frac{d}{dx'}$.

Thus

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu)=\frac{2k^2}{\sqrt{d}}\sum_{m=o}^{m=\infty}(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}(ikd)$$

$$\times\,P_n\left(\frac{d}{id(kx')}\right)\,P_m\left(\frac{d}{id\,(kx')}\right)\frac{\sin kr'}{kr'}$$

$$=\frac{2k^2}{\sqrt{d}}\sum_{m=o}^{m=\infty}(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}(ikd)\,\Sigma\,C_{m,\,n,\,p}\;P_p\left(\frac{d}{id\,(kx')}\right)\frac{\sin kr'}{kr'},$$

where $C_{m,\,n,\,p}=\dfrac{2p+1}{2}\displaystyle\int_{-1}^{1}P_m(z)\;P_n(z)\;P_p(z)\;dz$, and $p$ takes the values $n-m,\ n-m+2,\ n-m+4,\ \ldots\ n+m$, if $n>m$,
or $m-n,\ m-n+2,\ m-n+4,\ \ldots\ m+n$, if $m>n$.

Therefore

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\mu)=k\sqrt{\frac{2\pi k}{d}}\sum_{m=o}^{m=\infty}(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}(ikd)$$

$$\times\,\Sigma\,C_{m,\,n,\,p}\;i^p\,\frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_p(\mu')$$

$$=k\sqrt{\frac{2\pi k}{d}}\sum_{p=o}^{p=\infty}\frac{(2p+1)}{2}\,i^p$$

$$\times\,\frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}}\,P_p(\mu').\,\underset{a=0}{\text{Lim.}}\int_{-1}^{1}P_n(z)\;P_p(z)\left\{\sum_{m=o}^{m=\infty}e^{-m^2 a}(m+\tfrac{1}{2})\right.$$

$$\left.K_{m+\frac{1}{2}}(ikd)\;P_m(z)\right\}dz.\quad\Big]$$

Near the surface of the sphere B we have $r'<d$, so that by using the above expansion and by substitution in the boundary condition

$$\frac{d\psi_0}{dr}+\frac{d\psi_{\prime}}{dr'}=o\,,\ \text{ for } r'=b,$$

and on equating the co-efficient of $P_p(\mu')$ to zero, we at once obtain

$$A_p^{(1)}=-\left\{\frac{d}{db}\frac{J_{p+\frac{1}{2}}(kb)}{\sqrt{b}}\Bigg/\frac{d}{db}\frac{K_{p+\frac{1}{2}}(ikb)}{\sqrt{b}}\right\}\frac{2p+1}{2}\,i^p$$

$$\times \underset{a=0}{\mathrm{Lim.}} \int_{-1}^{1} \mathrm{P}_p(z)\, \phi(a, d, z) \times \left\{ \sum_{n=o}^{n=m} \mathrm{A}_n^{(o)}\, \mathrm{P}_n(z) \right\} dz,$$

and therefore

$$\psi_1 = -\sum_{p=o}^{p=\infty} \left\{ \frac{d}{db} \frac{\mathrm{J}_{p+\frac{1}{2}}(kb)}{\sqrt{b}} \Big/ \frac{d}{db} \frac{\mathrm{K}_{p+\frac{1}{2}}(ikb)}{\sqrt{b}} \right\} \frac{(2p+1)}{2} i^p$$

$$\times \frac{\mathrm{K}_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}}\, \mathrm{P}_p(\cos\theta')\, e^{ikct} \times \underset{a=0}{\mathrm{Lim.}} \left[ \int_{-1}^{1} \mathrm{P}_p(z)\, \phi(a, d, z) \right.$$

$$\left. \times \left\{ \sum_{n=o}^{n=m} \mathrm{A}_n^{(o)}\, \mathrm{P}_n(z) \right\} dz \right].$$

## § 5.

Now if we suppose that the vibrating sphere has a finite size then the scattered waves whose velocity potential we have determined in the last article will be incident on the vibrating sphere and will be again scattered. Let $\psi_2$ be the velocity potential of these last mentioned scattered waves regarded as radiating outwards from the centre of the sphere A. In determining the value of $\psi_2$, we ignore the presence of the sphere B.

Like $\psi_1$, $\psi_2$ has also to satisfy the conditions (i), (iii), and (iv) stated in § 3. The condition (ii) however for this case becomes

$$\frac{d\psi_1}{dr} + \frac{d\psi_2}{dr} = 0 \text{ for } r = a.$$

We can therefore assume $\psi_2$ to be given by the series

$$\psi_2 = \sum_{p=0}^{p=\infty} \mathrm{A}_p^{(2)} \frac{\mathrm{K}_{p+\frac{1}{2}}(ikr)}{\sqrt{r}}\, \mathrm{P}_p(\cos\theta)\, e^{ikct}.$$

Proceeding as in § 4, we find that

$$\mathrm{A}_p^{(2)} = -\left\{ \frac{d}{da} \frac{\mathrm{J}_{p+\frac{1}{2}}(ka)}{\sqrt{a}} \Big/ \frac{\mathrm{K}_{p+\frac{1}{2}}(ika)}{\sqrt{a}} \right\} \frac{(2p+1)}{2} i^p$$

$$\underset{a=0}{\mathrm{Lim}} \int_{-1}^{1} \mathrm{P}_p(z)\, \phi(a, d, z) \times \left\{ \sum_{n=0}^{n=\infty} \mathrm{A}_n^{(1)}\, \mathrm{P}_n(z) \right\} dz,$$

and therefore

$$\psi_2 = -\sum_{p=0}^{p=\infty} \left\{ \frac{d}{da} \frac{J_{p+\frac{1}{2}}(ka)}{\sqrt{a}} \Big/ \frac{d}{da} \frac{K_{p+\frac{1}{2}}(ika)}{\sqrt{a}} \right\} \frac{(2p+1)i^p}{2}$$

$$\times \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p(\cos\theta)\, e^{ikct}$$

$$\times \operatorname*{Lim.}_{a=0} \left[ \int_{-1}^{1} P_p(z)\, \phi(a, d, z) \left\{ \sum_{n=0}^{n=\infty} A_n^{(1)} P_n(z) \right\} dz \right].$$

## § 6.

We suppose that the scattered wave system determined by $\psi_2$ is again incident on the surface of the sphere B and is again scattered and that in this way an infinite number of reflections occur alternately on the two surfaces. The conditions satisfied by the velocity potentials of all these reflected wave systems are similar to those satisfied by $\psi_1$ aad $\psi_2$, and all these potentials are determined in a similar manner.

The complete solution of the problem is therefore given by the following series

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots + \psi_{2l} + \psi_{2l+1} \cdots,$$

where

$$\psi_{2l} = \sum_{p=0}^{p=\infty} A_p^{(2l)} \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p(\cos\theta).\, e^{ikct},$$

$$\psi_{2l+1} = \sum_{p=0}^{p=\infty} A_p^{(2l+1)} \frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\cos\theta').\, e^{ikct},$$

$$A_p^{(2l)} = -\left\{ \frac{d}{da} \frac{J_{p+\frac{1}{2}}(ka)}{\sqrt{a}} \Big/ \frac{d}{da} \frac{K_{p+\frac{1}{2}}(ika)}{\sqrt{a}} \right\} \frac{(2p+1)i^p}{2}$$

$$\operatorname*{Lim}_{a=o} \int_{-1}^{1} P_p(z)\, \phi(a,d,z) \times \left\{ \sum_{n=o}^{n=\infty} A_n^{(2l-1)} P_n(z) \right\} dz,$$

and $$A_p^{(2l+1)} = -\left\{ \frac{d}{db} \frac{J_{p+\frac{1}{2}}(kb)}{\sqrt{b}} \Big/ \frac{d}{db} \frac{K_{p+\frac{1}{2}}(ikb)}{\sqrt{b}} \right\} \frac{(2p+1)i^p}{2}$$

$$\operatorname*{Lim}_{a=o}\int_{-1}^{1} P_s(z)\,\phi(a,d,z)\times\left\{\sum_{n=o}^{n=\infty} A_n^{(2l)}\,P_n(z)\right\}dz.$$

## § 7.

I now take up the case when one sphere is inside the other and a vibration symmetrical about the line joining the two centres is prescribed either to the internal or the external sphere. I need only consider the case of two eccentric spheres, the case of two concentric spheres being very simple and the solution easily obtained by a direct method.*

The solution for this case also is easily obtained by adopting the method of successive reflections and proceeding in almost the same manner as in the case of two external spheres. The only difference in this case will be that Bessel's functions both of the first and the second kind will enter into the solution and that we shall have to use some other theorems similar to that enunciated in § 4.

For instance, suppose that the sphere A is inside the sphere B, and that we are required to determine the disturbance produced in the space inside the two spherical surfaces, when the prescribed vibration, expressible by the series

$$\sum_{n=o}^{n=m} A_n\,P_n(\cos\theta)\,e^{ikct},$$

is given to the inner sphere.

If $\psi_0$, $\psi_1$, $\psi_2$, etc., denote, as in the previous case, the velocity potentials of the initial unobstructed wave motion and the successive wave systems reflected alternately from the external and the internal surfaces, then

$$\psi_0 = \sum_{n=o}^{n=m} A_n \frac{\dfrac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}}{\dfrac{d}{da}\left[\dfrac{K_{n+\frac{1}{2}}(ika)}{\sqrt{a}}\right]}\,P_n(\cos\theta)\,e^{ikct}$$

$$= \sum_{n=o}^{n=m} A_n^{(o)}\,\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}}\,P_n(\cos\theta)\,e^{ikct},$$

---

* Lord Rayleigh, *Theory of Sound, Art 333.* Chree, *Messenger of Mathematics, Vol. XV, p. 20* (1886).

and if we bear in mind that in determining the velocity potential of any reflected wave-system from any surface we ignore the presence of the other surface, we can assume the following forms for $\psi_1$, $\psi_2$, $\psi_3$, etc.;—

$$\psi_1 = \sum_{p=0}^{p=\infty} A_p^{(1)} \frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\cos\theta')\, e^{ikct},$$

$$\psi_2 = \sum_{p=0}^{p=\infty} A_p^{(2)} \frac{K_{p+\frac{1}{2}}(ikr)}{\sqrt{r}} P_p(\cos\theta)\, e^{ikct},$$

$$\psi_3 = \sum_{p=0}^{p=\infty} A_p^{(3)} \frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\cos\theta')\, e^{ikct},$$

etc.

The unknown constants in each of these functions are easily determined from the boundary conditions which these functions satisfy, *viz.*,

$$\frac{d\psi_0}{dr'} + \frac{d\psi_1}{dr'} = 0 \text{ for } r'=b, \quad \frac{d\psi_1}{dr} + \frac{d\psi_2}{dr} = 0 \text{ for } r=a, \text{ etc.,}$$

by proceeding in a similar manner as in § 4, and by using the following two theorems*:—

(*a*) If $r^2 = r'^2 + d^2 - 2r'd\mu'$, then the function $\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\mu)$ can always be expanded as an infinite series of functions of the type $\frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\mu')$, provided $r' > d$.

We can write this theorem in the form

$$\frac{K_{n+\frac{1}{2}}(ikr)}{\sqrt{r}} P_n(\mu) = \sum_{p=o}^{p=\infty} B_{n,p} \frac{K_{p+\frac{1}{2}}(ikr')}{\sqrt{r'}} P_p(\mu'),$$

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[*] Both the theorems are easily proved by proceeding as in the theorem in § 4, if we notice that $\frac{e^{-ikr}}{r} = \frac{2}{\sqrt{dr'}}\sqrt{i} \sum_{n=o}^{n=\infty} i^n (n+\frac{1}{2}) J_{n+\frac{1}{2}}(kd) K_{n+\frac{1}{2}}(ikr') P_n(\mu')$,

$(r' > d)$

$$\frac{\sin kr}{r} = \frac{\pi}{\sqrt{dr'}} \sum_{n=o}^{n=\infty} (n+\tfrac{1}{2}) J_{n+\frac{1}{2}}(kd) J_{n+\frac{1}{2}}(kr') P_n(\mu')$$

(for all values of $r'$),

and $e^{-ikds} = \sqrt{\frac{\pi}{2kd}} \sum_{n=o}^{n=\infty} i^{3n}(2n+1) J_{n+\frac{1}{2}}(kd) P_n(s)$.

and we easily find that

$$B_{n,p} = (2p+1)\, k^2\, (-1)^p \int_{-1}^{1} P_n(z)\, P_p(z)\, e^{-ikdz}\, dz.$$

(*b*) Similarly, the function $\dfrac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}} P_n(\mu)$ can always be expanded as an infinite series of functions of the type $\dfrac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\mu')$.

Expressing this theorem in the form

$$\frac{J_{n+\frac{1}{2}}(kr)}{\sqrt{r}} P_n(\mu) = \sum_{p=0}^{p=\infty} C_{n,p} \frac{J_{p+\frac{1}{2}}(kr')}{\sqrt{r'}} P_p(\mu'),$$

we easily find that

$$C_{n,p} = (2p+1)\, k^2\, i^{p+n} \int_{-1}^{1} P_p(z)\, P_n(z)\, e^{-ikdz}\, dz$$

Using these two theorems, we find at once that

$$A_p^{(1)} = -\left\{ \frac{d}{db} \frac{K_{p+\frac{1}{2}}(ikb)}{\sqrt{b}} \Big/ \frac{d}{db} \frac{J_{p+\frac{1}{2}}(kb)}{\sqrt{b}} \right\} (2p+1)\, k^2\, (-1)^p$$

$$\times \int_{-1}^{1} P_p(z)\, e^{-ikdz} \times \left\{ \sum_{n=0}^{n=m} A_n^{(0)} P_n(z) \right\} dz,$$

$$A_p^{(2)} = -\left\{ \frac{d}{da} \frac{J_{p+\frac{1}{2}}(ka)}{\sqrt{a}} \Big/ \frac{d}{da} \frac{K_{p+\frac{1}{2}}(ika)}{\sqrt{a}} \right\} (2p+1)\, k^2\, i^p$$

$$\times \int_{-1}^{1} P_p(z)\, e^{-ikdz} \times \left\{ \sum_{n=0}^{n=\infty} i^n A_n^{(1)} P_n(z) \right\} dz,$$

etc.

## § 8.

In the foregoing articles I have restricted myself to the case when the prescribed vibration is symmetrical about the line joining the centres. When it is not symmetrical about the line joining the centres, the method adopted for the symmetrical case can easily be extended to this case with the help of three theorems (involving tesseral harmonics) similar to the three theorems which I have given for the symmetrical case.

# A note on the Stereoscopic representation of four-dimensional space.

BY

Syāmadas Mukhopadhyaya.

In an address "on the fourth dimension of space" delivered before the Moslem Istitute on the 3rd February, 1912, I referred to a stereoscopic device which had suggested itself to me, for visualizing figures in four-dimensional space. It may be mentioned that the possibility of visualizing four-dimensional figures has been predicted by Poincaré.

It is well known that a duplicate picture in plano of a solid figure, taken from two slightly different points of view, when properly looked at through a stereoscope, impresses one with the vividness of a single figure in three dimensions. On the same principle, suitably constructed wire diagrams in three dimensions, whose bases are stereoscopically related, should appear four-dimensional when viewed through a stereoscope. This will only hold for simple Geometrical figures about whose expected appearances in four-demensions the mind has been previously prepared by study and thought of four dimensional Geometry.

One simple experiment may be easily made by any one. Take a stereoscopic duplicate chart mounted on stiff card board, containing three white axes at right angles on a black back ground, some such charts as are enclosed with Airy's Tract on Partial Differential Equations. Stick a couple of equal white pins at the two origins normally to the board. If now the chart be viewed through a stereoscope, four white lines including the pin, will appear to stand out mutually at right angles, which is only possible in four-dimeusional space.

It should be observed that the picture on the retina is a two dimensional one. The effort at adjusting the optic axes of the eyes in binocular vision gives us the perception of a third dimension. The effort at contracting the crystalline lens for focussing at objects at near distances could give us the perception of a fourth dimension; but this latter adjustment takes place simultaneously with the former and not independently of it from acquired habit of looking at objects in three dimensions and consequently a certain amount of strain on the eyes is experienced when we try to realize through a stereoscope a four-dimensional figure. For the complete realization nevertheless we require more of mental development than organic.

# On a generalised force function of Painleve's type.

BY

HARIPROSANNA BANERJEE.

## *Introduction.*

According to the Principle of Lejeune-Dirichlet* we know that if the force-function at any point is a maximum the position is one of stable equilibrium. The converse of the theorem is assumed to be true in text-books on Dynamics, although it has never been rigorously demonstrated. Thus the criterion for the stability of equilibrium of a particle at a point is taken to be the existence of a maximum for the force-function at that point. But this is wrong as has been pointed out** by Professor Painlevé by a very simple example. The object of this paper is chiefly to generalise Painlevé's example as far as the present state of analysis permits. About the end of the paper, I have also investigated some force-functions for which, in the positions of stable equilibrium, the force has no differential coefficient and the force-function is not maximum.

I should like to express my indebtedness to Dr. Ganesh Prasad at whose suggestion I took up, and under whose guidance I carried on, the investigation the results of which are embodied in this paper.

## I.

It will be convenient for me, first to repeat the definitions of certain symbols to be employed by me in this paper, that were first introduced in the "Infinitär Calcul" of Du-Bois Reymond.*** Let us suppose that f and $\phi$ are two monotone functions of the continuous variable $x$ positive for all positive values of $x$, and let us consider the ratio $\frac{f}{\phi}$. As $x$ tends to zero (or to infinity), we may have three possibilities :

(a) $\frac{f}{\phi} \longrightarrow 0$, we write in this case, $f \prec \phi$;

(b) $\frac{f}{\phi} \longrightarrow \infty$, in this case, we have $f \succ \phi$;

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*Liouville's Journal*—Tome XII, 1847, p. 474—478.
See also Lejeune—Dirichlet's *Werke* Vol. II, p. 3—8.

***Comptes Rendus*, Tome 138, 1904.

***Untersuchungen Uber die Convergenz und Divergenz der Fourierschen Darstelulngsformeln." *Abhandlungen d. k. Akademie d. Wissenschaften zu Munchen*, 1876.

(c) $\frac{f}{\phi}$ tends to remain within two fixed limits and this is denoted by

$$f \asymp \phi ;$$

when this limit is finite, we have

$$f \sim A\,\phi .$$

By a logarithmico-exponential function or simply an *L*-function* is meant any real one-valued function which can be defined by a finite combination of the functional symbols,

$$\mathrm{A}(\cdots\cdots\cdots\cdots),\ \log\ (\cdots\cdots\cdots\cdots),\ e^{(\cdots\cdots\cdots)}$$

operating on the variable $x$ and on real constants.

As has been shown by Mr. Hardy ¶, any L-function is continuous and monotone for sufficiently small (or large) values of $x$; and one or other of the three relations

$$f \prec \phi, \quad f \succ \phi, \quad f \sim \mathrm{A}\phi$$

holds between any pair. Also under certain restrictions, the derivative of an *L*-function is an *L*-function, and the processes of differentiation and integration of *L*-functions are legitimate under these symbols.

## II.

Let us first consider a particle moving on the $x$-axis under a force $\frac{dU}{dx}$, the force function $U$ being defined by

$$U = \tfrac{1}{2} m\rho(x)\ \cos\ \sigma(x) \text{ or } o,$$

according as $x$ is different from, or is equal to zero, where $\rho\ (x)$ and $\sigma(x)$ are monotone functions of $x$; $\sigma(x)$ tends to infinity as $x$ tends to zero, and for values of $x$ different from zero, the first and the second differential coefficients of $\rho$ and $\sigma$ are existent, finite and continuous. Further, $\rho\ (x)$ is an even function of $x$.

I proceed now to study the conditions which $\rho$ and $\sigma$ must satisfy, so that $U$ may be a force-function of the required type.

## III.

1. First of all we have to ensure that the origin is regular for this value of $U$. For this, the condition is, that $U, \frac{dU}{dx}, \frac{d^2U}{dx^2}$ are all existent, finite and continuous at $x = o$.

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***Proceedings of the London Mathematical Society, Vol.* 10, 1912, page 54.

¶*Orders of Infinity* (Cambridge Mathematical Tracts, Vol. 10, 1910).

Since $U$ is continuous and zero for $x=o$, it is evident that $\rho(x)$ must also be continuous and zero for $x=o$.

Now,

$\left(\frac{dU}{dx}\right)_{x=o} = \lim_{x=o} \frac{\rho(x) \cos \sigma(x)}{x}$, and this limit should be zero in our case. Hence

$$\frac{\rho(x)}{x} \prec 1$$

$$\text{i.e.} \quad \rho(x) \prec x \qquad (a)$$

Again

$$\left(\frac{d^2 U}{d x^2}\right)_{x=0} = \lim_{x=0} \left\{ \frac{\rho'(x) \cos \sigma(x) - \rho(x) \sigma'(x) \sin \sigma(x)}{x} \right\},$$

and since there is neither maximum nor minimum at $x=0$, this must also be zero

Thus in the notation of the "Infinitär Calcul,"

$$\rho'(x) \cos \sigma(x) - \rho(x) \sigma'(x) \sin \sigma(x) \prec x.$$

Since $\cos \sigma(x)$ and $\sin \sigma(x)$ are oscillating functions, in the present state of analysis, the determination of the necessary and sufficient condition under which the above equation holds, is not possible.

So, we have to make each of the constituents zero.

Thus

$$\lim_{x=0} \frac{\rho'(x)}{x} \cos \sigma(x) = 0,$$

$$\text{and} \lim_{x=0} \frac{\rho(x) \sigma'(x) \sin \sigma(x)}{x} = 0,$$

Therefore

$$\left.\begin{array}{l} \rho'(x) \prec x \\ \rho(x) \sigma'(x) \prec x \end{array}\right\} \qquad (\beta)$$

If now both $\rho$ and $\sigma$ are $L$ — functions, the conditions $(a)$ and $(\beta)$ are much simplified.

Since $\rho'(x) \prec x$, we get by integration,

$$\rho(x) \prec x^2 \qquad (\gamma)$$

which includes $(a)$.

Again, the second of the conditions $(\beta)$ is

$$\rho(x) \sigma'(x) \prec x.$$

In connection with this condition, three cases arise.

**Case A** $\left(\sigma \prec \log \frac{1}{x}\right)$

$$\because\ \sigma' \prec \frac{1}{x},\ \rho(x)\,\sigma'(x) \prec x \quad \text{if}$$

$$\frac{\rho(x)}{x} \prec x,\ \textit{i.e.}\ \ \rho(x) \prec x^2,$$

and this is, the same as ($\gamma$).

In this case then, the conditions ($\alpha$) and ($\beta$) become

$$\underline{\rho(x) \prec x^2.}$$

### Case B $\left(\sigma \backsim A \log \frac{1}{x}\right)$

We may write

$$\sigma = b.\log\frac{1}{x} + \bar{\sigma} \ (\text{where } b \neq 0, \text{and } \bar{\sigma} \prec \log\frac{1}{x}).$$

$$\text{Then } \sigma' = -\frac{b}{x} + \bar{\sigma}' \backsim \frac{1}{x},$$

and hence $\rho(x)\,\sigma'(x) \prec x$

if $\rho(x) \prec x^2$.

In this case, as well, the conditions reduce to

$$\underline{\rho(x) \prec x^2.}$$

### Case C $\left(\sigma \succ \log\frac{1}{x}\right)$.

Here $\sigma' \succ \frac{1}{x}$, so the condition

$$\rho(x)\,\sigma'(x) \prec x$$

*involves* $\rho(x) \prec x^2$,

which is the condition ($\gamma$).

Thus, then in this case ($\alpha$) and ($\beta$) reduce to

$$\underline{\rho(x)\,\sigma'(x) \prec x.}$$

2. We have next to ensure that $U$ has no maximum at $x=0$.

Here $U - U_0 = \frac{1}{2}\, m\, \rho(x) \cos \sigma(x) \not< 0$,

for all values of x, satisfying the condition

$$0 < |x| < \epsilon,$$

$\epsilon$ being any sufficiently small positive quantity.

For, when $\sigma(x) = 2k\pi$.

($k$ being an arbitrary positive integer $> 0$),

$U = \frac{1}{2}\, m\, \rho(x) =$ positive,

and since $\sigma(x)$ tends to infinity as x tends to zero, these values of $\sigma(x)$

can be obtained in the immediate vicinity of the origin by making $k$ sufficiently large.

Thus $U$ has no maximum at the origin.

3. Thirdly, we have to ensure that at the origin we have stable equilibrium.

The equation of motion is,

$$m\frac{d^2 x}{d t^2} = \frac{d U}{d x}.$$

By integration we have,

$$\frac{m}{2}\left(\frac{d x}{d t}\right)^2 = U + \frac{m}{2}\left(\frac{d x}{d t}\right)^2_{x=0} - U(o)$$

$$= \frac{m}{2}\left\{\rho\,(x)\cos\sigma\,(x) + h\right\}$$

Let us take two points on the x – axis, $x_{+k}$ and $x_{-k}$, such that $x_{\pm k} = a_{\pm k}$, where $\sigma\,(a_k)$ is equal to $(2k+1)\,\pi$, k being an arbitrary positive integer $> 0$.

The above equation reduces to

$$\left(\frac{d x}{d t}\right)^2 = h - \rho\,(a_k).$$

If now $h \leq 0$, $\frac{d x}{d t}$ is imaginary, and thus x cannot depart from the interval $|\; x_{-k},\, x_k \;|$ which includes $x=0$. If $h > 0$, we can choose $a_k$ such that $\rho\,(a_k)$ is superior to $h$. Then, as well, x shall not depart from the said interval. Thus if $\delta$ be a small positive quantity chosen beforehand, x and $\frac{d x}{d t}$ (in the motion) are comprised between $+\,\delta$ and $-\,\delta$, so that the equilibrium is stable. Thus we see, that in all these cases, x is included in the interval which tends to be reduced to the origin.

4. Summing up the foregoing discussion, we arrive at the following conclusion: $\frac{1}{2}\, m\, \rho\,(x) \cos\sigma\,(x)$ *is a force-function of the required type provided that* $\rho\,(x) \prec x^2$ *when* $\sigma\,(x) \precsim \log\frac{1}{x}$ *or* $\rho\,(x)\,\sigma'\,(x) \prec x$ *when* $\sigma\,(x) \succ \log\frac{1}{x}$.

## IV.

### *Illustrative Examples.*

(1) $U = \frac{1}{2}\, m\, x^{\frac{14}{5}} \log\frac{1}{x^2} \cos\log\log\frac{1}{x}$.

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This comes under Case A

(2) $U = \frac{m}{2}\, x^{\frac{14}{5}} \log \frac{1}{x^2} \cos \log \frac{1}{x^2}.$

This comes under Case B.

(3)

(i) $U = \frac{1}{2}\, m\, x^4 \cos \frac{1}{x}$

(ii) $U = \frac{m}{2}\, e^{-\frac{1}{x^2}} \cos \frac{1}{x}$

(iii) $U = \frac{1}{2}\, m x^{2n+4} e^{-\frac{1}{x^{2n}}} \cos \underset{e}{\frac{1}{3x^{2n}}}.$

These examples come under case C which also includes Painlevé's example.

## V.

So long for the sake of simplicity we were considering the equilibrium of a particle moving on the axis of x. Let us now consider the more complicated case of three-dimensional space.

Let the force-function be defined by

$$U = \frac{m}{2}\left\{\rho_1 (x) \cos \sigma_1 (x) \pm \rho_2 (y) \cos \sigma_2 (y) \pm \rho_3 (z) \cos \sigma_3 (z)\right\}$$

where $\rho_1$, $\sigma_1$, etc., are functions of the variables as defined in § I, and where the sufficient conditions obtained therein are satisfied.

Thus the origin is a position of regular equilibrium, the force-function not having a maximum at the origin.

The equations of motion in this case are

$$m\frac{d^2x}{dt^2} = \frac{dU}{dx} = \frac{m}{2}\left\{\rho'_1 (x) \cos \sigma_1 (x) - \rho_1 (x)\, \sigma'_1 (x) \sin \sigma_1 (x)\right\} (1)$$

$$m\frac{d^2y}{dt^2} = \frac{dU}{dy} = \frac{m}{2}\left\{\rho'_2 (y) \cos \sigma_2 (y) - \rho_2 (y)\, \sigma'_2 (y) \sin \sigma_2 (y)\right\} (2)$$

$$m\frac{d^2z}{dt^2} = \frac{dU}{dz} = \frac{m}{2}\left\{\rho'_3 (z) \cos \sigma_3 (z) - \partial_3 (z)\, \sigma_3 (z) \sin \partial^3 (z)\right\} (3).$$

We can integrate these equations and find as before three intervals $\left|\overset{x}{-k_1}, \overset{x}{+k_1}\right|$, $\left|\overset{y}{-k_2}, \overset{y}{+k_2}\right|$, $\left|\overset{z}{-k_3}, \overset{z}{+k_3}\right|$ (the $k$'s being arbitrary positive integers $> 0$), all of them including the origin, such that the velocities parallel to the respective axes outside these intervals are imaginary. Hence the equilibrium is stable, for we can always describe a sphere of diameter equal to the smallest of these intervals, about the origin, and then $(x, y, z)$ cannot depart from the region bounded by the sphere.

## VI

Up to now, for the sake of simplicity, we have considered such force-functions that $U, \frac{dU}{dx}, \frac{d^2U}{dx^2}$ etc., are all continuous at the origin, but it is possible to take our force-functions such that $\frac{d^2U}{dx^2}$ is non-existent at the origin, and yet there is stability of equilibrium for $U$ at the origin.

In the preceding pages, while considering the force-function

$$U = \tfrac{1}{2} m \rho (x) \cos \sigma (x)$$

where $\sigma (x)$ tends to infinity as x tends to zero, I postulated that $\rho (x)$ and $\sigma (x)$ are monotone. I now proceed to consider some examples in which $\rho$ and $\sigma$ are not monotone :—

Ex. 1. $U = \frac{m}{2} x^2 \sin \frac{1}{x^k} \sin \frac{1}{x^h}$,

where $k$ and $h$ are either both odd or both even and $k \neq h$, $k$ and $h$ being both positive proper fractions.

Ex. 2. $U = \frac{m}{2} x^3 \sin \left( \frac{1}{\sin \frac{1}{x}} \right)$.

## VII.

In the foregoing articles I restricted myself to the discussion of such a force-function as has one of its factors a cosine function. A similar treatment is also applicable to functions in which instead of a cosine we have got either a sine or an sn-function.

Again, by the application of the Principle of the Condensation of Singularities* we can extend this peculiarity to all the points of an everywhere dense and enumerable aggregate.

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* See Hobson's *Theory of functions of a real variable*, Art. 438.

# * Quaternions, its origfn and development.

BY

A. C. Bose.

The symbol $\sqrt{-1}$ has played an important part in the history of mathematics. When we run our eye down the roll of illustrious mathematicians who have dealt with this mysterious symbol, we miss the names of our great Hindu mathematicians, Aryabhatta, Brahmagupta and Bhaskaracharya. We miss the names of the great Arabs such as Abu Zafar Muhammad Ibn Musa Al Khowarajmi whose name appears in the Latin "Algoritmi" which has passed at the present day into "algorism" or "algorithm." We miss the name of Omar Khayyam, astronomer, geometer and algebraist whose celebrated collection of quatrains, Fitzerald has made accessible to English readers. The first names which strike our eye, are those of the Italians—Cardan and Bombelli. We find that although they had worked with $\sqrt{-1}$, none of them had treated it otherwise than an ordinary algebraical symbol subject to the ordinary laws of algebra. We find also the great name of Napier of Merchistoun, who dealt with the square-roots of negative quantities in his De Arta Logistica but there is nothing in that fragment to show as has been remarked by George Chrystal, "how deeply the great inventor of logarithims had penetrated the secret which was to be hidden from mathematicians for 200 years". Familiar are the next three names on the roll,—those of Descartes, De Moivre and Euler. Descartes has rendered himself immortal by bringing about the union between Beautiful Analysis and Severe Geometry, the offspring of which union have upto the present day dominated the world of mathematics. To Leonhard Euler, "in whose voluminous writings" as Kelland has very justly said, "will be found a perfect store-house of investigations on every branch of algebraical and

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* Rowan Hamilton's own view of the calculus of Quaternions was once expressed as follows:—

"Time is said to have only *one dimension* and space to have *three dimensions*. The Mathematical Quaternion partakes of *both* these elements; in technical language it may be said to be 'time plus space' or 'space plus time' and in this sense it has or at least involves a reference to *four dimensions*. It shows

How the one of time, of space the Three
Might in the Chain of Symbols girdled be."

(*Vide* Graves's Life of Hamilton, Vol. 3, page 635).

mechanical science," we owe the first suggestion of the familiar $i$ for $\sqrt{-1}$, the Namkaran caremony, of the mysterious being born of the union of Analysis with Geometry. To him also we owe the formula

$$\cos\phi + i\sin\phi = e^{\phi i}$$

a formula which has played an important rôle in the province of complex algebra. To Abraham De Moivre, the French Mathematician who came to end his days in London, we owe the well-known formula

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$$

the significance of which in the province of universal algebra was understood much later on.

The name which arrests our attention as that of the discoverer of the pedigree of $\sqrt{-1}$ on the paternal side — the side of Geometry, is that of an Englishman—John Wallis, Savilian, Professor of Geometry at Oxford, one of the predecessors of James Joseph Sylvester, whose lectures on the principles of universal algebra, published two hundred years afterwards, have delighted the hearts of mathematicians like Willard Gibbs and taxed the abilities of lesser ones. Well known are Wallis's problem and Cayley's restatement of the same in the language of complex algebra. It may suffice to state that the idea of perpendicularity was connected by John Wallis with $\sqrt{-1}$.

Not so familiar as those I have already mentioned, is the name Caspar Wessel, a Danish mathematician, whose essay "Sur la representation de la direction" was presented to the Danish Academy in 1797 and published in 1799. This essay in which Wessel had not only dealt with an interpretation of the imaginary but also gave a quaternion theory with the help of the sphere and developed a complete spherical trignometry, remained unnoticod for nearly one hundred years, and was republished only in 1897.

Running our eye repidly down the roll we come to the Frenchman Abbe Buéé, who in a memoir printed in the Philosophical transactions of 1806, presented the same idea as that of Wallis, arrived at the law of composition of directed quantities in one plane, and considered a curve having branches in perpendicular planes, a subject afterwards much developed and generalised by Gregory Walton and George Salmon. We may note that Buéé's paper was read on June 20th, 1805 and Sir William Rowan Hamilton was born at Dublin on August 3rd, 1805 only a month and a half afterwards. We may note also that scarcely a year was over after the birth of Hamilton, when an opuscule was printed in Paris entitled "Essai Sur une Maniere de representer les Quantites Imaginaires dans les constructions Geometriques." The opuscule which did not bear the name of the author but gave the name of a certain lady of whom it could be had, remained unnoticed by mathematicians for nearly seven years when in Tome IV of Gergonne's Annals, there appeared a note by Français, an artillery officer in the city of Metz, dealing with the same subject. This led to a correspondence in the pages of the Annals, in which Francais, Argand, Gergonne and Servois took part. Francais admitted that he had his ideas from Argand, whose authorship of the celebrated opuscule became thus established. Through the influence of Cauchy and the sanction of the great Gauss, Argand's memoir has come to be regarded as classical on the subject and Hankel has justly claimed that Argand must be regarded as the true founder of the Geometrical representation of complex quantities in one plane.

John Warren and George Peacock, children of the English soil, upheld the reputation of the land of Wallis and Newton. In 1828, appeared Warren's Treatise on "the Geometrical representation of the square-roots of negative quantities"—a work which influenced much the future discoverer of Quaternions. In 1830 George Peacock, the Dean of Ely, one of that gallant band which set itself so resolutely and so successfully against the dotage of the Cambridge University in the ever memorable fight between D-eism and Dot-age, wrote his work on symbolical algebra in the preface to which, he characterised Warren's work "as distinguished for great originality and for the extreme boldness in the use of definitions." Warren gave a rule for Geometrical multiplication but we may note that Argand's method can at once be put in Warren's form and there is no doubt that Argand must be regarded as the inventor of Warren's rule of multiplication of directed quantities.

On the Continent the masterly exposition of complex numbers by Gauss in the preface to his second treatise on Biquadratic Residues, gained for them a triumphant entry into arithmetic operations. Well-known is the part played by the united couple $x+yi$ in the Gaussian plane, and also how the theory of orthomorphic correspondence (and of orthomorphic projection) of real points in the $z-$ and $Z-$ planes, was arrived at.

We have thus arrived at a definite and well-marked epoch in the history of $\sqrt{-1}$. Bhaskaracharya had declared that $\sqrt{-1}$ did not exist for the ordinary number system. He said "the square of a positive as well as of a negative number is positive and the square-root of a positive is double, positive and negative. There can be no square-root of a negative number, for this is no square." Bhaskara in the

12th century was right and he spoke as the arithmetician. The union of Geometry with Analysis was still 500 years in coming and we need not be sorry that in those early days it was not recognised that, fundamentally important was the conception that our common numbers did not form a complete system without the addition of the imaginary unit but that with the introduction of a second unit as Cayley so beautifully put it "numbers formed a universe complete in itself such that starting in it we were never led out of it."

In tracing the origin of Quaternions we must take note, next, of the valuable ideas introduced into Geometry by August Ferdinand Möbius in his work *Barycentric Calculus* (1827)—a work of which Clebsch said that "it can never be admired enough." The name of this great German is familiar to all lovers of Art in Mathematics. Folowing up a desideratum of Carnot's, Möbius used signs largely and defined for the first time the sign of a segment and even of an area. He succeeded further in extending the use of signs to lengths which are not measured on the same line, to angles which are not formed around the same point and to tetrahedra. The importance of this step can hardly be over-estimated. Thus, we all now know that:—

$$AB+BC+CA = 0$$

(A, B, C, being co-linear)

With D prefixed (D being a point outside the line)

$$DAB+DBC+DCA = 0$$

Again,

$$ABC - BCD+CDA - DAB = 0$$

(A, B, C and D being complanar points) with E, a point outside the plane, prefixed we have:—

$$EABC - EBCD+ECDA - EDAB = 0.$$

The similarity of these results to 'Multiplication of points' is apparent but Möbius left it to Grassmann of Stettin to discover it.

In the development of the complex algebra of a plane we find that the symbol $x+yi$ represents a point with $(x, y)$ co-ordinates. Connecting masses situated at points with the points themselves *i.e.* by combining magnitude with position, Möbius was led to the use of equations, expressing barycentric relation between the points. Thus, by writing the single equation

$(a+b+c+d)$ S$= a$A$+b$B$+c$C$+d$D, S being the centre of gravity of weights, $a,b,c,d$, placed at the points A,B,C,D, respectively.
Möbius by one step shook himself free of the trammels of co-ordinates and fixed his earnest gaze on the points themselves represented in the system by the single letters, A, B, C, D, &c. He subjected his equations

to the ordinary processes of multiplication and division by scalars and derived a host of useful and interesting results. His points may be considered as the "extraordinaries" of multiple algebra and it may be mentioned that he dealt also with the transformation of his fundamental points corresponding to the transformation of extra-ordinaries in multiple algebra. As Gibbs has said "it is a striking fact in the history of multiple Algebra, that the short period of less than two years was marked by the appearance of well developed and valuable Systems of multiple Algebra, by British, German and French authors, working apparently entirely independently of one another."—to wit—the Quaternions of Hamilton the Ausdeh nungslehre of Grassmann and the memoir of Saint Venant in Comptes Rendus, vol. 21, page 620.

Sir William Rowan Hamilton of Dublin, the future discoverer of quaternions had taken note of Argand's and Warren's investigations on complex quantities in a plane. But as their methods involved the conception of an angle (the argument), Hamilton considered if a clear interpretation and meaning could not be given to square-roots of negatives without introducing considerations so expressly geometrical as those which involve the conception of an angle. His thoughts thus began to run on a purely analytical basis, and the result was his paper published in 1835 on the "Theory of conjugate functions or algebraical couples with a preliminary and elementary essay on algebra as the science of pure time." (Transactiens of the Irish Academy Vol. 17, Part II, 1835). Sir William Rowan Hamilton was a close student of Kant who had thrown out in his Transcendental Æsthetic, the suggestion that it should be possible to construct, *ápriori*, a science of Time as well as a science of Space. Now Hamilton thought that as Geometry was the *ápriori* science of Pure space, (we should remember that these were pre-Riemann times), Algebra might be considered as a science of Pure time, *i.e.* as the science of order in progression. Thus, he came to concieve of moments of and steps in time and by introducing the ideas of a number-couple and of a step-couple arrived at the two formulae for products of couples :—

$$(1)\ (a_1\ a_2)\ (\alpha_1\ \alpha_2) = (a_1\alpha_1 - a_2\alpha_2,\ a_2\alpha_1 + a_1\alpha_2).$$

$$(2)\ (a_1\ a_2)\ (b_1\ b_2) = (a_1b_1 - a_2b_2,\ a_2b_1 + a_1b_2)$$

when $(a_1\ a_2)$ in equation (1) is a number-couple and $(\alpha_1, \alpha_2)$ is a step-couple while in (2) both $(a_1\ a_2)$ and $(b_1\ b_2)$ are number-couples. It was then easy to show by putting $\alpha_2 = 1$, $\alpha_1 = 0$ and by operating twice with the number couple (0, 1) that

$(0, 1)^2 = -1$ or $(0, 1) = \sqrt{-1}$, (Arts 14 and 17 in the Preface to Lectures on Quaternions, 1853), In this way Hamilton avoided the use of expressly geometrical considerations used by Argand and Warren in arriving at a conception of $\sqrt{-1}$ which did not involve "anything obscure, impossible or imaginary." To those, who like Cayley, did not appreciate, algebra as the science of Pure time, the argument may be put thus :—

If x, y, &c.... denote ordinary (real or imaginary) analytical magnitudes which are susceptible of addition and multiplication, and for which the commutative and associative laws hold good, we may consider a multiple symbol (x, y.......) susceptible of addition and multiplication according to determinate laws. To fix ideas, let us take the couples (x, y), $(x', y')$. *One* law of addition would be

$$(x, y) + (x', y') = (x+x', y+y')$$

so that as regards addition, the multiple symbols are commutative and associative. *One* law of multiplication would be

$$(x, y)\ (x', y') = (x x'-yy', xy'+yx')$$

so that as regards multiplication, the multiple symbols are commutative and associative.

If we assume next that if $k$ be any ordinary analytical magnitude, then

$$k\,(x, y\ .\ .\ ) = (kx, ky, ..\quad)$$

Then clearly the couple (x, y) can be set as $(x, y) = x\,(1, 0) + y\,(0, 1)$ and if we set $(1, 0) = i$ and $(0, 1) = j$

$$(x, y) = xi+yj$$

The letters i, j, which thus stand for the definite multiple symbols (1, 0), (0, 1) are termed by Cayley "extraordinaries" to distinguish them from ordinary analytical magnitudes—real or imaginary.

Next, taking the assumed law of multiplication of couples to be

$$(x, y), (x', y')=(xx'-yy', xy'+yx')$$

we find that

$$\begin{aligned}(ix+jy), (ix'+jy') &= i(xx'-yy')+j(xy'+yx')\\ &= i^2xx'+ijxy'+jiyx'+j^2yy'\end{aligned}$$

[by the distributive law which, it is implied, holds good]. We, thus, at once obtain the laws of combination of the extraordinaries by which binary products are reduced to linears *viz.*—

$$i^2=i\,;\ ij=ji=j\,;\ j^2=-i$$

we see that $i$ behaves like unity and may be put$=1$ and we may speak of the extraordinaries as 0, 1, $j$ where $j^2=-1$. Instead of taking the couple $(x, y)$, we might have taken the triplet $(x, y, z,)$; $(x, y, z)$ would then be expressible as

$$xi+yj+zk \text{ where } i=(1, 0, 0)\ j=(0, 1, 0), \text{ and } k=(0, 0, 1).$$

*And by an assumed law of multiplication* we might arrive at the laws of combination of the extraordinaries $i$, $j$ and $k$ and vice versa. *One* such law may be

$$(x, y, z)(x', y', z')=(yz'-y'z, zx'-z'x, xy'-yx')$$

which would be associative but not *commutative.*

Hamilton having arrived at his laws of couples, endeavoured to extend them to triplets and higher multiplicities called sets. The results at which he arrived are remarkable and may be considered to be an anticipation of the Linear associative algebras which were to come thirty-five years afterwards. Hamilton confined his attention, however, to triplets, to use his own words "with the desire to connect in some new and useful (or at least interesting) way calculation with Geometry, through some undiscovered extension to space of 3 dimensions, of a method of construction or representation which had been employed with success by Mr. Warren (and Argand) for operation on right lines in one plane." But when it was proposed to leave the plane and to construct a system which should have some general analogy to the known system of Argand and Warren but should extend to space, difficulties of a new character arose in the endeavour at surmounting which, Hamilton was encouraged by his friend, Mr. John T. Graves, Professor of Jurisprudence in the University College in London. Between the two intimate friends and in the seclusion of their studies, various systems of triplets were devised but none proved satisfactory. Thoughts however are never lost, much less the thoughts of mighty brains. They act and react and at last produce definite results. The crux of the problem was to conceive a law of multiplication for triplets which would be distributive and associative and if possible commutative and would produce a triplet. In a letter written shortly before his death by Hamilton to his son Archibald he wrote "every morning in the early part of October 1843, on my coming down to breakfast, your (then) little brother William Edwin and yourself used to ask me, "well, papa, can you multiply triplets," whereto I was obliged to reply with a sad shake of the head, "No, I can only add and subtract them." In 1843, then, Hamilton had resumed "the endeavours to adopt the general conception of triplets to the multiplication of lines in space, resolving to retain the distributive principle and at first supposing that he could preserve the commutative principle also." On 16th October 1843, Sir William, who was President of the Royal Irish Academy, was walking in to attend a council meeting, and although he talked to his wife who was with him, from time to time, "an undercurrent of thought was going on in his mind" as regards the multiplication of triplets "which gave at last a result whereof he felt at once the importance." "An electric circuit seemed to close" "and a spark flashed forth, the herald as he foresaw immediately, of many long years to come, of definitely directed thought and work, by himself if spared and at all events on the parts of others." Hamilton could not resist the impulse to cut with a knife on a stone of Brougham Bridge the fundamental formulae

$$i^2=j^2=k^2=ijk=-1$$

which contained the solution of the Problem he had been meditating upon. He pulled out on the spot a pocket-book which still exists and made an entry on which, at the very moment he felt that, it might be worth his while to expend the labour of at least 10 or 15 years to come. Hamilton felt a problem to have been solved—an intellectual want relieved, which had haunted him for at least 15 years before. Less than an hour elapsed before Hamilton asked for and obtained leave of the Council to read at the next general meeting of the Academy a paper on quaternions which was done on November 13, 1843.

Three score years and twelve have passed and the carvings on Brougham Bridge have mouldered away. *Sic transit gloria mundi*; but not so easily pass away the thoughts and joys and hopes of men. And the joy of a mathematician treading "the silent desert of a great New thought," is divine and lives for ever. Such joy as Hamilton experienced must have been experienced by Newton when he saw the apple to fall—by Adams and Leverrier when the telescope of Galle swept the heavens and discovered Neptune,—by Faraday when he conceived of the lines of force traversing all space—by Maxwell when he arrived at his electro-magnetic theory—by Gauss when he solved the problem of biquadratic residues—by Grassmann when he arrived at his conception of combinatorial products—by Cayley when he conceived of the Absolute connecting metrical and projective properties of space—by SophusLie when he came to conceive of the Infinitesimal transformation and by Willard Gibbs when he produced his Statistical mechanics. Hamilton has laid bare his soul in his letters to Graves, De Morgan and Tait and we see clearly the workings of a mighty intellect, its doubts and difficulties and shortcomings and, above all, its candour and humility.

It appears that on the day following the discovery, Hamilton wrote a letter to his friend J. T. Graves, the sharer of his troubles and the enthusiastic supporter of his endeavours, giving a graphic account of the birth of i, j, k, which would repay perusal and is given below :—

"My dear Graves—A very curious train of mathematical speculation occurred to me yesterday, which I cannot but hope, will prove of interest to you. You know that I have languished to possess a theory of Triplets analogous to my published theory of couplets and also to Mr. Warren's Geometrical representation of Imaginary Quantities. Now I think that I discovered yesterday a theory of Quaternions, which includes such a theory of Triplets."

"My train of thoughts was of this kind. Since $\sqrt{-1}$ is, in a certain well-known sense, a line perpendicular to the line 1, it seemed natural that there should be some other imaginary to express a line perpendicular to both the former and because the rotation from 1 to this also, being doubled, conducts to—1, it also ought to be a square-root of negative unity, though not to be confounded with the former. Calling the old root, as the Germans often do, i and the new one j, I enquired what laws ought to be assumed for multiplying together, $a+ib+jc$ and $x+iy+jz$.

It was natural to assume the product

$$=ax-by-cz+i\ (ay+bx)+j\ (az+cx)+ij\ (bz+cy)$$

But what are we to do with i j ? Shall it be of the form $a+i\beta+j\gamma$ ?

Its square would seem to be=1, because

$$i^2=j^2=\ -1.$$

and this might tempt us to take—

$$ij=1 \text{ or } ij=\ -\ 1.$$

But with neither assumption shall we have the sum of the squares of the coefficients of 1, i, and j in the product=to the product of the corresponding sums of squares in the factors." Take the simplest case of a product *viz.* the case when it becomes a square; we have

$$a^2-b^2-c^2+2iab+2jac+2ijbc \text{ and } (a^2-b^2-c^2)^2+(2ab)^2+(2ac)^2$$

$$=(a^2+b^2+c^2)^2$$

∴ the condition respecting moduli is fulfilled, if we suppress term involving i j altogether, and what is more, $a^2-b^2-c^2$, $2ab$, $2ac$ are precisely the co-ordinates of the *square* point, so to speak, deduced from the point a, b, c in *space* by a slight extension of Mr. Warren's Rule for points on a plane. In fact, if we double, in its own plane, the rotation from the positive semi-axis of x to the radius vector of the point a, b, c, we obtain the direction of the radius vector to

$$a^2-b^2-c^2,\ 2ab,\ 2ac.$$

Behold me, therefore, tempted for a moment to fancy i j=0. But this seemed odd and uncomfortable and I perceived that the same suppression of the term which was *de trop*, might be attained by assuming what seemed to me less harsh *viz.* j i$=-$ i j. I made $\therefore$

$$ij=k, ji=-k$$

reserving to myself to inquire whether k was$=0$ or not.

For this purpose, I next multiplied $a+ib+jc$ (multiplier) into $x+ib+jc$ (multiplicand) keeping still, as you see, the two factor lines in one commen plane with the unit line. The result was

$$ax-b^2-c^2+i(a+x)b+j(a+x)c+k(bc-bc)$$

in which the coefficient of k still vanishes and $ax-b^2-c^2$, $(a+x)b$, and $(a+x)c$ are easily found to be the correct co-ordinates of the *product-point*, in the sense that the rotation from the unit line to the radius vector of a, b, c, being added in its own plane to the rotation from the same unit line to the radius vector of the other factor-point $x$, b, c conducts to the radius vector of the lately mentioned product-point and that this latter radius vector is in length the product of the two former. Confirmation of ij$=-$ji but no confirmation yet of the value of k. Try, boldly, then the general product of two triplets and seek whether the law of the moduli is satisfied when we suppress the k altogether.

$$\text{Is } (a^2+b^2+c^2)(x^2+y^2+z^2)\equiv(ax-by-cz)^2+(ay+bx)^2+(az+cx)^2\ ?$$

No ! the first member exceeds the second by $(bz-cy)^2$. But this is just the square of the co-efficient of k in the development of the product $(a+ib+jc)(x+iy+jz)$ if we grant $ij=-ji=k$ as before. And here there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triplets or transferring the paradox to algebra, must admit a *third* distinct imaginary symbol k, not to be confounded with i or j but equal to the product of the first as multiplier and the second as multiplicand ; and therefore was led to introduce *quaternions* such as

$$a+ib+jc+kd \text{ or } (a, b, c, d).$$

I saw that we had probably $ik=-j$ because $ik=i\,ij$ and $i^2=-1$ and that in like manner we might expect to find $kj=ij\,j=-i$ from which I thought it likely that $ki=j$ and $jk=i$, because it seemed likely that if $ji=-ij$ we should have also $kj=-jk$ and $ik=-ki$.

And since *the order of multiplication of these imaginaries is not indifferent*, we cannot infer that

$k^2=ij.ij$ is $+1$ because $i^2\times j^2=-1\times-1=+1$. It is more likely that

$$k^2=ij.ij=-i.i.jj=-1$$

And in fact this last assumption is necessary, if we would conform the multiplication of quaternions to the law of the multiplication of the moduli.

My assumptions were now completed *viz.*—

$$\text{(A)}\quad \begin{aligned} & i^2=j^2=k^2=-1 \\ & ij=-ji=k \\ & jk=-kj=i \\ & ki=-ik=j \end{aligned}$$

and with these I was obliged to assume that if

$$(a, b, c, d,).\quad (a', b', c', d')=(a'', b'', c'', d'').$$

then the four following equations of multiplication hold good.

$$\text{(B)}\quad \begin{aligned} & a''=aa'-bb'-cc'-dd' \\ & b''=ab'+ba'+cd'-dc' \\ & c''=ac'+ca'+db'-bd' \\ & d''=ad'+da'+bc-b'c. \end{aligned}$$

But I considered it essential to try whether these equations were consistent with the law of Moduli *viz.*—

$$\sum a''^2=\sum a^2\times\sum a'^2.$$

It can be seen that this is so.

We have then the 1st law for the multiplication of two quaternions together *viz.* the law of Moduli. Division of Quaternions is easy. The equations (B) of multiplication give

$$\text{(C)}\quad \begin{aligned} & a'=(a^2+b^2+c^2+d^2)^{-1}(aa''+bb''+cc''+dd'') \\ & b'=(a^2+b^2+c^2+d^2)^{-1}(-ba''+ab''+dc''-cd'') \\ & c'=(a^2+b^2+c^2+d^2)^{-1}(-ca''+ac''+bd''-db'') \\ & (d'=(a^2+b^2+c^2+d^2)^{-1}(-da''+ad''+cb''-bc''). \end{aligned}$$

Adition and subtraction require no notice

Making $a=\eta\cos\theta$, $b=\eta\sin\theta\cos\phi$ $c=\eta\sin\theta\sin\phi\cos\psi$ and $d=\eta\sin\theta\sin\phi\sin\psi$. I would call $\theta$ the *amplitude* of the Quaternion, $\phi$ its co-latitude and $\psi$ its longitude. Its modulus is $\eta$. Representing the three co-efficients $b$, $c$, $d$ of the imaginary triplet $ib+jc+kd$ by the rectangular co-ordinates of a point in space, $\eta\sin\theta$, as being the radius vector of this point, might be called the radius or perhaps the *length* of the Quaternion. We may speak of the inclination of one Quaternion to another and its Cosine is

$$\text{Cos}\,\phi\cos\phi'+\sin\phi\sin\phi\cos(\psi'-\psi)=\frac{bb'+cc'+dd'}{\eta\eta'\sin\theta\sin\theta'}.$$

If with the amplitudes, $\theta$, $\theta'$, $\theta''$ of any two factors and their product, as sides we construct a spherical triangle, the angle opposite to the amplitude of the product will be the supplement of the inclination of the factors to each other and the angle opposite to the amplitude of either factor will be the inclination of the other factor to the product."

We thus see that a remarkable result had been arrived at and that the analogy with the product of two complex quantities in two dimensions in that the law of moduli was satisfied, was complete. We see also that the discovery had an *analytical* basis. Analysis has always been a prolific mother. Newton's Principia severely geometrical in its final shape arose out of Analysis,—Poncelet's geometrical system rested on an analytical basis and we know from the publication of the Saratoff manuscripts that the principles which served as a foundation for the *Traite des proprietes Projectives*, were established by the aid of the Cartesian analysis. On Hamilton's part the geometrical conception of a Quaternion as a ratio of two vectors in space followed quickly. And in the Cambridge and Dublin Mathematical Journal, Vol. I (1846) were published under head "Geometry of 3 dimensions" three unfinished papers "on Symbolical Geometry" presenting the geometrical aspect of Quaternions. It has been said, with justice, that "Hamilton first *invented* the quaternion and then *discovered* it. The invention consisted in the conception of the three units $i, j, k$ with their special laws of combination and in the construction of an assosciative algebra with four fundamental units. The quaternion was then discovered to be a complex number representing the ratio of two vectors in space and the ruling doctrine of the calculus is that "$S\alpha\beta + V\alpha\beta$ is a quantity which however it may operate on or taken in conjunction with, a like quantity, gives rise to a quantity of the same analytical nature."

This discovery, unlike the equally memorable discoveries of Grassmann of Stettin made in a more extensive field, independently end almost at the same time on the continent, which remained unnoticed for over 20 years, took the mathematical world by storm. Men like Spottiswoode De Morgan, and Herschell became its early students. Arthur Cayley who, in 1845 gave the best testimony of the value he attached to it by operating with the new method, called it "a very beautiful theory" and "a very interesting discovery," although in after years he held an adverse opinion with Lord Kelvin as regards its utility (vide pp. 152-165 in C. G. Knott's Life and Scientific Work of Tait). De Morgan who had written on both Double and Triple Algebra admitted that Hamilton had the 'right sow by the ear' and that any system which had to do with rotations must drop the commutative law. Thirty years afterwards when Hamilton had long passed away Clerk] Maxwell wrote:—

"But for many purposes in physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian co-ordinates and to fix the mind at once on a point of space instead of its 3 co-ordinates, on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other, although the ideas connected with it did not receive their full development till Hamilton made the next great step in dealing with space by the invention of the Calculus of Quaternions......I am convinced that the introduction of the ideas as distinguished from the operations and methods of Quaternions will be of great use in electro-dynamics the relations of which can be expressed far more simply by a few words of Hamilton's than by the ordinary equations." The reader may also consult C. G. Knott's life and work of Tait for some more views of Maxwell. Professor Clifford, who studied at the close of his brief but glorious life, the extensive calculus of Grassmann and expressed his high admiration for it, said in a letter to Professor Sylvester on a complex algebra for space of any number of dimensions "it thus appears that Quaternions are the last word of geometry in regard to complex algebras."

Hamilton matured his ideas for ten years, lectured upon the calculus from time to time and published in 1853 his Lectures on Quaternions—a book which has been unfortunately allowed to go out of print, for, it has a charm all its own. In this work Hamilton developed the Quaternions from the point of view of ratio of two directed quantities, introduced the selective terms. Scalar, Tensor, Versor and Conjugate, established their laws, proved that all the laws of Scalar algebra except the commutative (the key-note of the calculus) held good for Quaternions, dealt with the powers of Vectors (the exponents being Scalars) and showed that they were quaternions. Hamilton proved that Quadrantal versors obeyed the same laws as vectors (thereby meeting the apparently serious objection that his i, j, k are made to do double duty, *viz.* of quadrantal versors as well as unit vectors), gave an interpretation of q.y, when the vector, was not in the plane perpendicular to the axis of the quaternion q, established the law of the conical operator $q(\quad)q^{-1}$, laid down the equations, in the language of the calculus, of the straight line, the plane, the conics and the conicoids, expressed Macullagh's Modular generation of surfaces of the second order in the language of quaternions, expounded his beautiful theory of the Linear vector function (which has been developed by Tait from the point of view of homogeneous strains), explained the meaning of the word "differential" in the calculus on the lines of Newton's celebrated Lemma, extended Taylor's series to quaternions, applied the processes to curves and surfaces, to the theory of curvature and to physical Astronomy, explained what was involved in the conception of a definite integral in quaternions, dealt with the differentiation of implicit functions; explained the theory of quadratic equations in quaternions and what was meant by biquaternions (a totally different conception was attached afterwards to it by Clifford). Truly, then, the calculus Hamilton invented was a marvellous one and is worth serious study.

Hamilton's 'Elements of Quaternions' is a more finished work and was published after the great mathematician had passed away. Being conscious of the imperfections of his Lectures as a treatise, Hamilton intended to write a Manual of 400 pages but the work grew in his hands and although it had reached 700 pages, it was not brought to a close and death cut him off in the midst of his labours. I will not trouble my readers with a description of the arrangement and plan of the more finished, work, for, the late Professor Joly, Hamilton's successor in the chair of astronomy at Dublin, has earned the gratitude of quaternionists by bringing out a second edition of the elements with additions of his own. Mr. William Edwin Hamilton, Hamilton's son who had several conversations with his father on the subject of the Elements, shortly before his death, has told us that "his father spoke of anticipated applications of quaternions to electricity and to all questions in which the idea of *polarity* was involved—applications which he never in his own life-time expected to be able fully to develop —bows to be reserved for the hands of another Ulysses." Joly has told us that Hamilton intended to have added some account of the operator V and an appendix containing notes on Anharmonic Co-ordinates and on the Barycentric calculus. We are thankful to Lieutenant Hime for having recently written a book on Anharmonic Co-ordinates based on Hamilton's ideas..

After the death of Hamilton the quaternionic mantle fell on the worthy shoulders of Professor Guthrie Tait, whom Hamilton himself designated "as eminently fitted to carry on, happily and usefully this new branch of mathematical science and likely to become in it one of the chief successors of its inventor." Faithfully and well has the great disciple fulfilled his mission. He has enriched the science with original investigations as regards physical applications of the highest importance *viz.* Kinematics of a deformable system, Kinetics of a rigid system, geometrical and physical optics, electro-dynamics, and Hydrokinetics; and Tait's applications of the operatorV to Line, Surface, and Volume integrals, to magnetic problems, to the stress function, and in connection with the Calculus of variations, perhaps won him from Maxwell the pleasing title, "chief musician upon Nabla." As the late Alexander Macfarlane said "In future times Tait will be best known for his work in the Quaternion analysis. Had it not been for his expositions, developments and applications, Hamilton's invention would be to-day in all probability a mathematical curiosity" In addition to his treatise on quaternions and other papers in various journals, Tait wrote in collaboration with Professor Kelland an excellent elementary treatise on the principles of the science, a third edition of which has been prepared by Dr. C. G. Knott of the University of Edinburgh, another ardent student of the science who has tilted many a lance under the banner of Tait with Professors Willard Gibbs and Oliver Heaviside in the memorable fight in the early nineties of the past century over the vexed question of the utility of quaternions in physics and allied subjects. I cannot pass without notice Mr. A McAulay, another student of the science who has contributed largely to the physical applications of quaternions. In addition to his little treatise on the "utility of quaternions in physics" he contributed a paper in the Philosophical Magazine on Quaternions as a practical instrument of physical research." Sir Robert Ball's Theory of screws and Professor Clifford's Biquaternions may in a certain sense, be called developments of Hamilton's Quaternions. The late Professor C. J. Joly, Hamilton's successor in the chair of astronomy, has materially advanced the science by contributions extending its physical applications, and by giving a new interpretation to the quaternion *viz.*. by considering it capable of representing a definite point loaded with a definite weight or mass, has given an interesting treatment of projective geometry by quaternions. In his book "Manual of quaternions" published in 1905, he has shown the fundamental simplicity of the science by giving its whole grammar in the first 20 pages and in the last chapter has given a sketch of the properties of associative algebras applicable to hyper or *n*— dimensional space. I cannot omit to notice also, the remarkable paper by Homersham Cox in Volume XIII of the Transactions of the Cambridge Philosophical Society on the extension of the quaternions and the Ausdehnungslehre (Grassmann's extensive calculus) to different kinds of uniform space. These have removed to a certain extent the sarcastic lament of Tait that 'Quaternions were not applicable to hyper space and that they were contented to deal with those poor three dimensions in which mere mortals were doomed to dwell but which cannot bound the limitless aspirations of a Cayley or a Sylvester.' The Appendix to Joly's edition of the Elements is a rich contribution by the late professor C. J. Joyly on (I) quaternion determinants (ii) miscellaneous properties of two linear vector functions (iii) the strain function (iv) Invariants of linear vector functions (v) on the general linear transformation in space (vi) on the theory of screws

(vii) on finite displacements (viii) on the Kinematical treatment of curves [an extension of the method employed by Hamilton in Art. 396 of his Elements] (ix) on the Kinematical treatment of surfaces (x) on systems of rays and (xi) on the operator $\nabla$—all showing the power and scope of Hamilton's grand calculus.

Quaternions considered as a branch of Mutiple algebra have been developed in the classical memoir of Benjamin Pierce on Linear associative algebras, which was circulated privately in 1870 and which has been made accessible to the public by Pierce's worthy son, Charles Pierce, with valuable notes and additions in Volume IV of the American Journal of mathematics. This Linear associative algebra can, according to Cayley, be made the common analytical basis of the various geometrical theories of earlier and recent dates. Sir William. Hamilton's quaternions coincide with a quadruple algebra of Pierce's systems in which division is unambiguous. On the analytical aspect of quaternions, Cayley's chapter in Tait's Quaternions is a valuable addition. There Cayley has shown the intimate connection of Quaternions with Matrices. Hamilton's exquisite theory of the Linear Vector function really involves the Theory of Matrices, the exposition of which by Cayley about the middle of the 19th century Sylvester, in his own inimitable language called "the ushering in of the reign of algebra the Second" To the student of physics the theory is of particular importance in the mathematical treatment of phenomena connected with nonisotropic media.

Tait and Joly are dead and the judgment of the mathematical world has not been invariably favourable to the great analyst's discovery. But we should remember that in the memorable fight between Vector analysis and Quaternions, sufficient note was not taken of the incontestable fact, as pointed out by James Byrnie Shaw, the author of the valuable synopsis of Linear associative algebras, that "a quaternion including a scalar and a vector as particular quaternions is a Number Hamilton extended our number system into the broader domain of Linear associative algebra and gave us the only extension in which division remains possible. The dust raised by contending parties over the question of what systen of algebraic symbols and methods would be most useful in applied mathematics has some times obscured the fundamental facts in the case of quaternions. But all this aside, the fact remains that real quaternions is the next extension of algebra beyond ordinary complex numbers. The field which will be developed in coming years, as the field of the Theory of functions of a complex variable has been developed in the 19th century, will be that of the functions of quaternions." We already hear of a Hamiltonian group in the all unifying group theory showing that Hamilton's genii i, j, k, have not been idle.

As regards the question of judging of the value of a mathematical system by the scope and extent of its applications to physics, I may quote from a letter of Jacobi to Legendre "It is true that M. Fourier believed that the main aim of mathematics was public utility and the explanation of natural phenomena but a philosopher of his ability ought to have known that the sole aim of science is the honour of the human intellect and that on this ground a problem in numbers is as important as a problem on the system of the world". We all hope that the "dust of contending parties" has now been laid and the production of the great Irishman will now have an impartial treatment. An International Association was formed at Dublin for promoting the study of quaternions and allied systems of mathematics and Dr. Felix Kllin was at the head of a commission to investigate the relative merits of different methods. Those in terested in the subject I would refer to the *Bulletin ef the intermational association for promoting the Study of Quaternions and alliap systems of mathematics*for June 1913, containing an appreciation of the labours of the late Dr. Alexander Macfarlane, some time President of the Association. Alexander Macfarlane's is an honoured name in Vector analysis,—and to him the object of the Association was "The promotion of sound thinking with regard to this young branch of mathematics and the unification of points of virew on the basis of sound Philosophical principles". His last paper on vector Analysis, as Generalied Algebra was read before the Cambridge congress of mathematicians in 1912.

I have endeavoured to sketch above only the origin and development of Quaternions, feebly at best, and I can not venture to sum up here the voluminous literature which has been published on the allied Systems. Might I be permitted to hope to write on "Quaternions and after" in a future issue of our Bulletin?

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13. Reale Accademia dei Lincei di Roma.
14. Königliche B. Akademie der Wissenschaften zu Munchen.
15. Royal Irish Academy.
16. American Mathematical Society.

# Notes and News.

The Academy of Sciences of Paris will award the following prizes in 1917:—

The Bordin Prize of 3,000 Franks for perfecting, in some important point, the arithmetical theory of non-quadratic forms.

The Vaillant Prize of 4,000 Franks for determining and studying all the surfaces which may be generated, in two different ways, by the displacement of an invariable curve.

With reference to Mr. Mazumdar's paper in the last issue of the Bulletin, Mr. G. R. Kaye has sent the following note:—

I am much obliged to the Honorary Secretary for kindly allowing me to see, before publication, Mr. Mazumdar's interesting paper on Āryabhaṭa. As Mr. Mazumdar points out there are several misprints in my note of 1908.

Āryabhaṭa's text itself is by no means unambiguous and my original translation (used by Mr. Mazumdar) does not satisfy me. Conclusions based upon such a text are very liable to contain ambiguities.

I still hold that the fundamental process involved in Āryabhaṭa's rule is contained in Euclid (Books ii and vii) but this does not necessarily mean that the rule was a direct development from Euclid. The point is that nowhere in Indian works do we meet with any process at all that could lead up to the rule.

I agree with Mr. Mazumdar that the introduction of an *assumed quantity* is important: this conception played a somewhat significant part in later Indian mathematics and would, possibly, be worth further investigation from the historical point of view.

I venture to draw attention to my note on *The Two Aryabhaṭas* in *Bibliotheca Mathematica* (Leipzig) 1910 (p. 239 f.). There is certainly an element of doubt as to the authenticity of the text and it behoves one to be cautious in building upon it. My later views on this subject will be found in *Indian Education* (August 1910 p. 16 f,) and in my *Indian Mathematics* recently published by Messrs. Thacker, Spink & Co